JOURNAL OF COMPUTATIONAL PHYSICS 21, 340–342 (1976)

Note

Solutions to Divergence Form Equations Using the Method of Partial Implicitization

The original development of partial implicitization is given in [1] and was shown in [2] to be well suited for use on vector-processing computers since the method is an explicit unconditionally stable numerical technique. This method has recently been modified from its original second-order accurate form to achieve fourth-order accuracy, see [3]; however, all of the developments to date have made for equations cast in nondivergence form. As shown in [4], equations cast in the divergence form are to be preferred because of their inherently greater accuracy. The success (unconditional stability) of the method of partial implicitization depends directly on the nondivergence form of the equations to be solved. Thus a problem arises of how to apply partial implicitization to divergence form equations. Since the method of partial implicitization is applicable only to relaxing problems to their steady state (method is not applicable to true transient problems), a simple mathematical manipulation can be carried out which will produce the desired partial implicitization solution to divergence form equations.

Using Burgers' equation as a model equation (see [1, 4, 5]), the partial implicitization solution to divergence form equations can be demonstrated. The nondivergence form of Burgers' equation is

$$\frac{\partial U}{\partial t} + \left(U - \frac{1}{2}\right)\frac{\partial U}{\partial \eta} - \nu \frac{\partial^2 U}{\partial \eta^2} = 0 \tag{1}$$

and the divergence form is

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial}{\partial \eta}(U^2 - U) - \nu \frac{\partial^2 U}{\partial \eta^2} = 0. \tag{2}$$

As discussed in [6], the linearized von Neumann stability condition for equations in nondivergence form is the same as for the divergence form. With this important fact, Eq. (1) can be modified to achieve both unconditional stability and divergence form by adding the term

$$\left(U - \frac{1}{2}\right)\frac{\partial U}{\partial \eta} - \frac{1}{2}\frac{\partial}{\partial \eta}(U^2 - U)$$

to the right-hand side of Eq. (1). Before doing so, the following time related quantities are defined: superscript $N+1$ is the new (advanced) time level at which a solution is to be obtained and superscript N is the current time level at which all quantities are known. Eq. (1) becomes

$$\frac{\partial U}{\partial t} + \left(\left(U - \frac{1}{2}\right)\frac{\partial U}{\partial \eta} - \nu \frac{\partial^2 U}{\partial \eta^2}\right)^{N+1} = \left(\left(U - \frac{1}{2}\right)\frac{\partial U}{\partial \eta} - \frac{1}{2}\frac{\partial}{\partial \eta}(U^2 - U)\right)^{N}. \quad (3)$$

In the von Neumann stability analysis, the right-hand side of Eq. (3) is identically zero and does not enter the stability analysis. Thus Eq. (3), when put in partial implicitization form as in [1], is unconditionally stable. In addition Eq. (3) in the steady-state limit does, in fact, conform to the desired divergence form

$$\frac{1}{2}\frac{\partial}{\partial \eta}(U^2 - U) - \nu \frac{\partial^2 U}{\partial \eta^2} = 0. \quad (4)$$

Solutions to Eqs. (1) and (3) were obtained using the method of partial implicitization as given in [1]. These solutions were obtained over the range $-5 \leqslant \eta \leqslant 5$ for $\nu = \frac{1}{8}, \frac{1}{16}$, and $\frac{1}{24}$. The boundary conditions were

$$U = 1 \text{ @ } \eta = -5$$
$$U = 0 \text{ @ } \eta = 5.$$

With the above boundary conditions, the solution of Burgers' equation represents a shock wave centered about the point $\eta = 0$. The initial conditions used to begin the calculations are:

$$\begin{aligned} U &= 1 \quad \text{for } -5 \leqslant \eta \leqslant -h \\ U &= 0.5 \quad \text{for } \eta = 0 \\ U &= 0 \quad \text{for } h \leqslant \eta \leqslant 5. \end{aligned}$$

A range of step sizes ($h = \Delta\eta$) were run and the error between the exact solution and the partial implicitization solution was calculated as

$$\bar{E} = \frac{\sum_{i=1}^{M} | U_i - U_{e,i} |}{M}$$

where M is the total number of interior points and U_e is the exact solution. The results are given in Fig. 1 where it is immediately obvious that the divergence form is, as expected, always more accurate (by approximately a factor of 3) than the nondivergence form. The slopes of the curves in Fig. 1 range from a minimum of 1.97 to a maximum of 2.05 indicating second-order accuracy as would be expected from the use of equally spaced central finite differences.

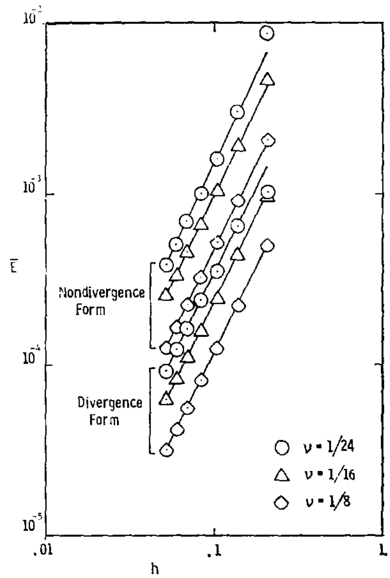

FIG. 1. Average errors from divergence and nondivergence form solutions to Burgers' equation.

Thus by adding the nondivergence terms and subtracting the divergence terms on the right-hand side of the nondivergence equation, a divergence form solution can be obtained in an unconditionally stable manner using partial implicitization.

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RECEIVED: October 15, 1975.

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